Mother templates for gravitational wave chirps

B.S. Sathyaprakash

Department of Physics and Astronomy, Cardiff University, Cardiff, UK

Abstract. Templates used in a search for binary black holes and neutron stars in gravitational wave interferometer data will have to be computed on-line since the computational storage and retrieval costs for the template bank are too expensive. The conventional dimensionless variable $T = (c^3/Gm)t$, where m is the total mass of a binary, in the time-domain and a not-so-conventional velocity-like variable $v = (\pi Gmf)^{1/3}$ in the Fourier-domain, render the phasing of the waves independent of the total mass of the system enabling the construction of *mother templates* that depend only on the mass ratio of a black hole binary. Use of such mother templates in a template bank will bring about a reduction in computational costs up to a factor of 10 and a saving on storage by a factor of 100.

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Gravitational waves from binary black holes during the last few seconds of their inspiral and merger are good candidates for a first direct observation by interferometric gravitational wave detectors presently under construction. The computational costs of generating and storing templates used in a search for these waves is quite expensive. Conventionally, one has used a two-parameter family of templates corresponding to the two masses m_1 and m_2 of the component stars in a binary (equivalently, one has also used the total mass $m = m_1 + m_2$ and the (symmetric) mass ratio $\eta = m_1 m_2/m^2$ or the Newtonian and post-Newtonian *chirp times,* etc.). However, as is well-known, general relativity allows us to use a dimensionless time-variable $T = (c^3/Gm)t$ in terms of which all dynamical equations become independent of the total mass but depend only on the mass ratio η. Therefore, relativists need only study a one-parameter family of black hole binaries of different mass ratios; the conclusion they draw will be applicable for binaries of different total masses. Of course, in doing so one has to use an appropriate re-scaling of all physical scales.

In gravitational wave data analysis one has so-far not taken advantage of this scaling since: (1) one deals with the detector data acquired in the real time t and not the adimensional time T and (2) one requires for the purpose data analysis samples $h_k \equiv h(t_k)$, $t_k \equiv k\Delta t$, that are equally spaced in an apriori chosen time-interval Δt, corresponding to a sampling rate $f_s = 1/\Delta t$. It is obvious that the samples $h_K \equiv h(T_K)$, $T_K \equiv K\Delta T$, equally spaced in T yield the required samples h_k only for a binary whose total mass is an integral multiple of $m = c^3 M_c/G$, where M_c is some characteristic mass. In this article we show that a one-parameter family of samples $h_K(\eta)$, supplemented

with a simple linear interpolation, are sufficiently accurate for the construction of a two-dimensional search template bank $h_k(m,\eta)$. This brings about a saving on computational costs by a factor of up to 10 and storage costs by a factor 100.

In the rest of this article we work with units $c = G = 1$.

1. Time-domain waveform

Post-Newtonian (PN) theory computes the gravitational wave (GW) flux $F(v)$ emitted, and the (dimensionless) relativistic binding energy $E(v)$ of, a compact binary as expansions in the gauge independent velocity v in the system. For a binary consisting of two non-spinning black holes, of masses m_1 and m_2, in quasi-circular orbit about each other, the flux and energy functions are both known to order v^5 (i.e. 2.5 PN) beyond the quadrupole approximation [1, 2]:

$$\begin{aligned} F(v) = \frac{32\eta^2 v^{10}}{5} \Bigg[1 - \left(\frac{1247}{336} + \frac{35}{12}\eta \right) v^2 + 4\pi v^3 \\ + \left(-\frac{44711}{9072} + \frac{9271}{504}\eta + \frac{65}{18}\eta^2 \right) v^4 - \left(\frac{8191}{672} + \frac{535}{24}\eta \right) \pi v^5 + O(v^6) \Bigg], \end{aligned} \tag{1}$$

$$E(v) = -\frac{\eta v^2}{2} \left[1 - \left(\frac{9+\eta}{12} \right) v^2 - \left(\frac{81 - 57\eta + \eta^2}{24} \right) v^4 + O(v^6) \right]. \tag{2}$$

Gravitational waves are dominantly emitted at twice the orbital frequency of the system. The GW frequency is related to the invariant velocity v via $v = (\pi m f_{\rm GW})^{1/3}$. In the quasi-circular, adiabatic approximation one uses the energy balance between the flux of gravitational waves lost from the system and the rate of decay of the binding energy of the system: $F(v(t)) = -m(dE/dt)$.

The energy balance equation can be used to set up a pair of coupled, non-linear, ordinary differential equations (ODEs) [3] to compute the orbital phase evolution $\varphi(t)$ of the binary during the adiabatic regime:

$$\frac{d\varphi}{dt} - \frac{v^3}{m} = 0, \quad \frac{dv}{dt} + \frac{F(v)}{mE'(v)} = 0, \tag{3}$$

where we have made use of the relation between the orbital frequency and the GW frequency, viz, $f_{\rm orb}(t) = \dot\varphi(t)/(2\pi) = f_{\rm GW}(t)/2 = v^3(t)/(2\pi m)$ and $E'(v)$ denotes the v-derivative of $E(v)$: $E'(v) = dE/dv$. The above differential equations allow us to compute a two-parameter family of phasing formulas $\varphi_k^{m,\eta}$, $t_k = k\Delta t$, corresponding to the parameters (m,η), of the binary that helps us to construct search templates. Until now this has been the method of computing and storing templates. However, the cost of generating and storing templates in this way is quite expensive.

In the *restricted* PN approximation, where the amplitude of the waveform is kept to the lowest PN order, while the phase is expanded to the highest PN order known, the GW radiation emitted by a binary at a distance r from Earth and sensed by an interferometric antenna, is described by the waveform

$$h(t) = \frac{4C\eta m}{r} v^2(t) \cos[\phi(t)], \tag{4}$$

where $\phi(t) = 2\varphi(t)$ is the gravitational wave phase. C is a constant that takes values in the range $[0, 1]$ depending on the relative orientation of the source and the antenna. It has an r.m.s (averaged over all orientations and wave polarisations) value of $2/5$.

Let us introduce a dimensionless time-variable $T = t/m$. In terms of this new variable the differential equations in equation (3) take the form

$$\frac{d\varphi}{dT} - v^3 = 0, \quad \frac{dv}{dT} + \frac{F(v)}{E'(v)} = 0. \tag{5}$$

The solutions of these two differential equations would yield a phasing formula $\varphi(T; \eta)$ that depends, unlike the phasing formula $\varphi(t; m, \eta)$, on only one parameter, namely the mass ratio η. The advantage of this new phasing formula is that for the purpose of data analysis it is sufficient to create and store a finely sampled one-parameter family of *mother templates* φ_K^η, where $T_K = K\Delta T$. The phasing formula for a binary of a given total mass m and mass ratio η, namely $\varphi_k^{m,\eta}$, can be computed from the mother template, supplemented by a linear interpolation, without having to solve the ODEs all over again:

$$\varphi_k^{m,\eta} = (1-x)\varphi_K^\eta + x\varphi_{K+1}^\eta, \quad K = [k/m], \quad x = k/m - K. \tag{6}$$

where for any real number f, $[f]$ denotes the integer closest to, but smaller than, f. This way of computing templates saves on computational costs by a factor of between 5 (for $T-$approximants) and 10 (for P-approximants), depending on the signal model, and storage costs by a factor of 100 (the number of templates in the 'm-coordinate').

In solving the differential equations above one must specify the initial conditions, namely the orbital phase and velocity at a chosen instant of time, say the instant of coalescence. These will, of course, depend on the total mass of the system. However, by choosing the instant of coalescence to be zero one can make any explicit dependence on the total mass vanish.

2. Frequency-domain waveform

We shall show in this Section that the result we found in the previous Section for the time-domain phasing formula holds good in the frequency domain too.

The stationary phase approximation to the Fourier transform of the waveform in equation (4) is given by [4, 6, 7]

$$\begin{aligned}\tilde{h}(f) &\equiv \int_{-\infty}^{\infty} h(t) e^{2\pi i f t} dt \\ &= \frac{4C\eta m}{r} \int_{-\infty}^{\infty} \frac{v^2(t)}{2} \left[e^{i\psi_f^+(t)} + e^{i\psi_f^-(t)}\right] dt \\ &\simeq \frac{2C\eta m}{r} \frac{v^2(t_f)}{\sqrt{\dot{f}_{\rm GW}(t_f)}} e^{i[\psi_f(t_f) - \pi/4]},\end{aligned} \tag{7}$$

where $\psi_f^\pm = 2\pi f t \pm \phi(t)$, an overdot denotes a derivative w.r.t. t and t_f is the stationary point of the Fourier phase, $\psi_f^-(t)$ when $f \geq 0$ and $\psi_f^+(t)$ when $f \leq 0$, namely $\dot{\psi}_f^\pm(t_f) = 0$. Indeed, t_f turns out to be the instant when the GW frequency $f_{\rm GW}$ is numerically equal

to the Fourier frequency $f : f = \dot{\phi}(t_f)/(2\pi) = f_{\rm GW}(t_f)$. One can solve for t_f by inverting the PN expansion of $f_{\rm GW}(t)$.

On substituting for the stationary point t_f, and consistently using the available PN expansions of the flux and energy functions one gets the usual stationary phase approximation (uSPA) to the inspiral signal

$$\psi_f(t_f) = 2\pi f t_C - \phi_C + 2\int_{v_f}^{\infty} dv (v_f^3 - v^3)\frac{E'(v)}{F(v)} = 2\pi f t_C - \phi_C + \Psi(f; m, \eta) \tag{8}$$

where $v_f \equiv v(t_f) = (\pi m f)^{1/3}$. t_C is the time of coalescence and ϕ_C is the phase at $t = t_C$, both of which will be set to zero. Just as in the time-domain, the frequency-domain phasing is also given, in the SPA, by a pair of coupled, non-linear, ODEs:

$$\frac{d\psi(f)}{df} - 2\pi t(v_f) = 0, \quad \frac{dt(v_f)}{df} + \frac{m(\pi m)^{1/3}}{3}\frac{E'(v_f)}{F(v_f)}\frac{1}{f^{2/3}} = 0. \tag{9}$$

From the computational point of view, solving the ODE's above is more efficient than computing the phase algebraically using equation (8). For most binaries the uSPA is sufficiently accurate. One, therefore, generates the signal directly in the Fourier domain using equation (9). This is a lot quicker than solving the ODEs in equation (3) and then Fourier transforming. As in the time-domain, one can reduce the seemingly two-parameter family of differential equations in equation (9) to a one-parameter family by using $v = (\pi m f)^{1/3}$ as a Fourier variable instead of f and a dimensionless 'time function' $T(v) = t(v)/m$. In that case the resulting equations are

$$\frac{d\psi(v)}{dv} - 6v^2 T(v) = 0, \quad \frac{dT(v)}{dv} + \frac{E'(v)}{F(v)} = 0. \tag{10}$$

3. Explicit fast frequency-domain signal models

We shall now give explicit formulas for the frequency-domain signal in v-representation for T-approximants. As for P-approximants it is best to use the differential equations in equation (10).

$\Psi(f)$ can be found by substituting in equation (8) for the flux and energy from equations (1) and (2), re-expanding the ratio $E'(v)/F(v)$ and integrating term-by-term. The resulting Fourier phase can be conveniently expressed as $\Psi(f; m, \eta) = \sum_0^4 \Psi_k(f)\tau_k$, where Ψ_k are functions only of the Fourier frequency and τ_k are the so-called (dimensionless) PN *chirp times* given in terms of the binary mass parameters

$$\tau_0 = \frac{5}{256\eta v_0^5}, \ \tau_1 = 0, \ \tau_2 = \frac{5}{192\eta v_0^3}\left(\frac{743}{336} + \frac{11}{4}\eta\right), \tau_3 = \frac{\pi}{8\eta v_0^2},$$
$$\tau_4 = \frac{5}{128\eta v_0}\left(\frac{3058673}{1016064} + \frac{5429}{1008}\eta + \frac{617}{144}\eta^2\right), \tag{11}$$

with $v_0 = (\pi m f_0)^{1/3}$, f_0 is a fiducial frequency (e.g., the lower cutoff of the antenna response) and the Ψ_k are given, in terms of the scaled frequency $\nu \equiv f/f_0$, by:

$$\Psi_0 = \frac{6}{5\nu^{5/3}}, \ \Psi_2 = \frac{2}{\nu}, \ \Psi_3 = \frac{-3}{\nu^{2/3}}, \ \Psi_4 = \frac{6}{\nu^{1/3}}. \tag{12}$$

The amplitude in (7) is a simple power-law of the form $f^{-7/6}$ [4].

The new representation that can speed up signal generation is obtained by working in the Fourier domain with the post-Newtonian expansion parameter v. Introduce a velocity-like variable v. Indeed, v is nothing but the parameter v_f in equation (8). Use v as the Fourier variable instead of f. On substituting $f = v^3/(\pi m)$, and setting the instant of coalescence t_C and the phase at ϕ_C to zero, we find

$$\Psi(f) = \sum_{k=0}^{4} \theta_k v^{k-5}, \tag{13}$$

where the *chirp parameters* θ_k are given by

$$\theta_0 = \frac{3}{128\eta},\ \theta_1 = 0,\ \theta_2 = \frac{5}{96\eta}\left(\frac{743}{336} + \frac{11}{4}\eta\right), \theta_3 = -\frac{3\pi}{8\eta},$$
$$\theta_4 = \frac{15}{64\eta}\left(\frac{3058673}{1016064} + \frac{5429}{1008}\eta + \frac{617}{144}\eta^2\right). \tag{14}$$

The Fourier transform in equation 7 can now be written as

$$\tilde{h}(f) = \frac{Cm^2}{r}\sqrt{\frac{5\pi\eta}{384v^7}} \exp\left[i\sum_{k=0}^{4} \theta_k v^{k-5} - i\pi/4\right], \tag{15}$$

Note that in this form the phase depends only on the mass ratio η and not on the total mass m[9]. One can take advantage of this feature in lowering the computational costs in generating templates in the following manner: One computes a set of look-up Tables, or mother templates, $\Psi(v;\eta)$ vs v for different values η. The Fourier phase $\Psi(f;m,\eta)$, for a binary of mass m and mass ratio η, can simply be read off from the appropriate look-up table: $\Psi(f;m,\eta) = \Psi(v = (\pi m f)^{1/3};\eta)$. Moreover, the cost of storing a one-parameter set of tables of $\Psi(v;\eta)$ should be several orders of magnitude lower than the cost of storing a two-parameter family of Tables $\Psi(f;m,\eta)$.

4. Practical Considerations

The codes for generating mother templates have been implemented and tested for accuracy and are available for a free-download [5]. We find that the waveforms generated by the interpolation method agree well with the waveforms directly computed using the ODEs. However, certain practical matters should be kept in mind while using these codes. Firstly, it is most accurate to work with either of the time- or frequency-domain phases, $\varphi(T)$ or $\Psi(v)$, respectively, rather than the corresponding waveforms. In other words, one must compute and store a mother phasing formula and *never* a mother waveform. Interpolating waveforms is much harder, and more inaccurate, than the phase of a waveform, which is a monotonic smooth function. Secondly, the initial velocity v_0 of a mother template required in solving the differential equations for the phasing $\varphi(t)$ must correspond to the binary of lowest total mass $m_{\rm min}$ in a search: $v_0 = (\pi m_{\rm min} f_0)^{1/3}$, where f_0 is the lower frequency cutoff of a detector. Thirdly, if the sampling interval in real time is Δt then the sampling interval ΔT in adimensional time should be chosen equal to $\Delta t/m_{\rm max}$, where $m_{\rm max}$ is the maximum mass of in a search, in

order for the interpolating formula to work best. The last two points can be summarised as: *the smallest velocity in a mother template should correspond to the lightest binary in a search and the adimensional sampling rate should correspond to the heaviest binary.* This means that if the real sampling rate is f_t, then in a search for binaries of masses $m < 10M_\odot$ one must choose a sampling rate $f_T \simeq 5\times 10^{-2}(f_t/1kHz)(m_{\max}/10M_\odot)$ [10].

Finally, the integration of the ODEs must be terminated at, or just before, the time when the system reaches the last stable orbit defined by the velocity at which the energy function has an extremum, that is $E'(v_{\rm lso}) = 0$. The right had side of the equation for velocity evolution in equation (5) diverges at this point and integration cannot be continued beyond $v = v_{\rm lso}$. At certain post-Newtonian orders and approximations the velocity evolution close to, but well before, the last stable orbit is not monotonic; these correspond to the case where the flux function $F(v)$ has a zero before the last stable orbit velocity. In those cases one must stop the integration before reaching $dv/dT = 0$, to avoid the code from crashing.

5. Conclusions

The use of an adimensional time, well-known in the literature on black hole binaries, in the time-domain, or a velocity-like parameter in the Fourier-domain, allows us to introduce a one-parameter family of *mother* templates which significantly reduces the computational costs of generating and storing binary inspiral search templates that will be used in the analysis of data from the up-coming ground-based gravitational wave interferometers. Mother templates of binary inspiral waveforms would also speed-up data analysis simulations that require a large number of waveforms of different masses.

It is not clear at the present time what implications this representation will have for the actual choice of search templates. It is conceivable that one will be able to define a 'v-transform' of the data so that the problem becomes a one-parameter search for binary black holes with different mass ratios. However, let us not forget that such a scheme will only be able to detect those signals whose instant of coalescence is $t_C = 0$; in other words only those that coalesce at the end of a given data segment. One will, therefore, have to explicitly search for binaries with every possible instant of coalescence, as opposed to the presently planned scheme wherein one takes advantage of the fact that one can search for (almost) all instants of coalescence, in one-go, by working in the Fourier-domain. A new scheme, therefore, is unlikely to significantly change the computational costs of a search, although the method may have certain advantages over the conventional approach. Whether or not an alternative method based on the mother templates introduced in this work is yet to be seen.

Mother templates that are independent of the total mass of a binary will be available at all post-Newtonian orders, in the effective one-body approach [8, 3] and in the case of eccentric and spinning binaries too. Their use should make it easier for us to incorporate new parameters, such as eccentricity and spins, in our search templates to detect compact binaries in gravitational wave data.

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[9] Although the amplitude does depend on the total mass, it is irrelevant in template generation.

[10] The sampling rate in adimensional time is a factor $m_{\max}/m_{\min}$ larger than the minimal required sampling rate. This could be a large factor requiring very large memory to hold each mother template. However, one can circumvent the problem by working with a small number of mother templates for each mass ratio.